Experimental observation of phase synchronization

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An experimental observation of phase synchronization is presented for two unidirectionally coupled chaotic Rössler systems. We show that in this case phase synchronization is connected with generalized synchronization which occurs when the coupling strength exceeds a critical value.[S1063-651X(96)03408-3]

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The synchronization properties of uni- or bidirectionally coupled chaotic systems currently is a topic of active research [1–8]. Numerous investigations have shown that there exist not just a single way how nonlinear systems may oscillate in synchrony but different degrees of synchronization. The strongest notion of synchronization requires that the difference of the state vectors of the coupled systems converges to zero in the limit $t\to\infty$. This definition is most widely used and will be called *identical synchronization* (IS) in the following. On the other hand it is also possible that the state vectors of the coupled systems are (asymptotically) related by some (possibly complicated) function [9–11]. This type of synchronization is called *generalized synchronization* (GS) and occurs, for example, with uni-directionally coupled systems if the response system is a passive system (i.e., it possesses only negative conditional Lyapunov exponents) [11]. Another example of synchronization is the so-called *phase synchronization* (PS) that has been discovered for chaotic systems only recently by Rosenblum *et al.* [12]. In order to describe this phenomenon a suitable phase variable has to be defined for the systems of interest. This can be done heuristically for strange attractors that spiral around some particular point (or hole) in state space like the Rössler attractor shown in Fig. 1. In such a case, a phase angle $\phi(t)$ can be defined that de- or increases monotonically. Phase synchronization of two coupled systems occurs if the difference $|\phi_1(t)-\phi_2(t)|$ between the corresponding phases is bounded by some constant [13]. Using the phase angle $\phi(t)$ one may define a mean rotation frequency

$$\Omega=\lim_{t\to\infty}\frac{\phi(t)}{t}. \tag{1}$$

In the case of PS, this mean rotation frequency is the same for the drive and the response system, i.e., also for chaotic systems PS leads to the frequency entrainment known from coupled periodic oscillations. Note that PS does not imply IS, i.e., the amplitudes of both systems can be completely uncorrelated [12].

In the following we present an experimental observation of PS and show that in this case PS is a consequence of GS. Furthermore, we show how PS manifests itself in the power spectrum. The experimental system consists of two unidirectionally coupled Rössler systems (2) and (3) that have been implemented on an analog computer:

$$\begin{aligned}\alpha\dot{x}_1&=2+x_1(x_2-4),\\ \alpha\dot{x}_2&=-x_1-\omega_1x_3,\\ \alpha\dot{x}_3&=\omega_1x_2+0.412x_3,\end{aligned}\tag{2}$$

$$\begin{aligned}\alpha\dot{y}_1&=2+y_1(y_2-4),\\ \alpha\dot{y}_2&=-y_1-\omega_2y_3,\\ \alpha\dot{y}_3&=\omega_2y_2+0.412y_3+c(x_3-y_3).\end{aligned}\tag{3}$$

The parameters $\omega_1=1$ and $\omega_2=1.1$ determine the mean rotation frequency around the center of the attractors of the drive and the response system, respectively, and $\alpha=0.013$ gives the parametrization of time due to the hardware of the analog computer. The attractors have been reconstructed from time series (16 bit resolution, 1 kHz sampling frequency) of the x_2 and the y_2 variable using the method of delays and are shown for $c=0$ in Fig. 1. From these reconstructions the mean rotation frequencies have been extracted based on time series of 32 k lengths. For the case of no coupling given in Fig. 1 both rotation frequencies are different due to the different parameters $\omega_1=1$ and $\omega_2=1.1$ in Eqs. (2) and (3). This difference still exists for sufficiently small values of the coupling parameter c as can be seen in

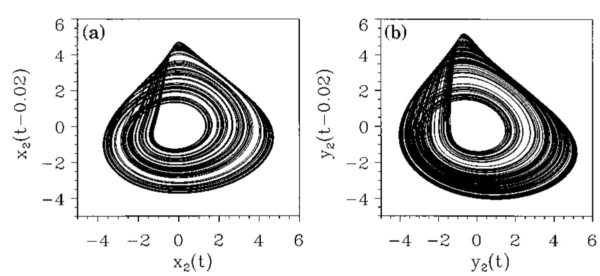

FIG. 1. Delay reconstruction of the attractors of the drive (a) and the response system (b) given by Eqs. (2) and (3), respectively. Both time series were generated experimentally using an analog computer. The mean rotation frequencies are $\Omega_1=11.82$ Hz (a) and $\Omega_2=13.62$ Hz (b).

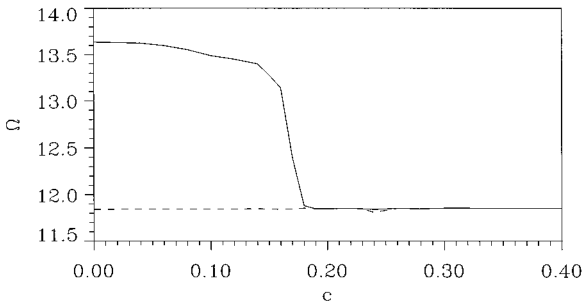

FIG. 2. Mean rotation frequencies Ω_1 (dashed) and Ω_2 (solid) in Hz vs the coupling parameter c. For $c>0.18$ phase synchronization occurs and both rotation frequencies coincide.

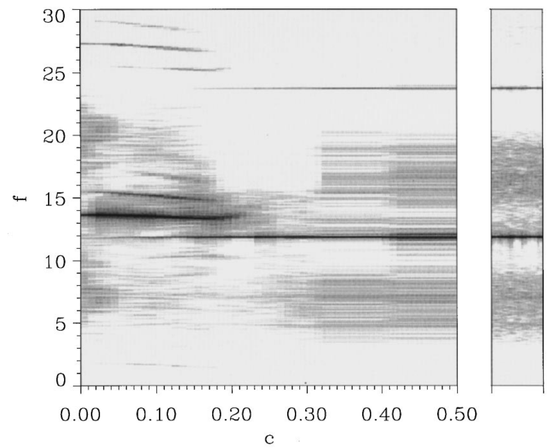

FIG. 4. Power spectrum (frequency axis vertical in Hz, amplitudes gray scaled) of the response system (left) and the drive (right) vs coupling parameter c.

Fig. 2 where the mean rotation frequencies Ω_1 (dashed) and Ω_2 (solid) are plotted vs c. At $c\approx0.18$ the response system undergoes a transition to a new phase synchronized state where the mean rotation frequencies of the drive and the response coincide.

Figure 3 shows the phase difference $\Delta\phi=\phi_1-\phi_2$ as a function of time for different values of the coupling constant c. For small coupling ($c=0.1$) $\Delta\phi$ increases almost linearly in time. As soon as PS occurs $\Delta\phi$ undergoes a bounded chaotic oscillation ($c=0.2$).

Phase synchronization can also be observed in the power spectrum of the response system as shown in Fig. 4. The main spectral components given by the dark lines move to the dominant frequencies of the drive that are shown in the power spectrum plotted in the bar on the right-hand side. The frequency entrainment is clearly visible, but the threshold value for PS cannot be uniquely determined from the spectrum.

We shall now discuss the mechanism underlying the phenomenon of phase synchronization. Consider again the equations (3) of the response system. If the coupling constant c is increased starting from zero a new term $-cy_3$ is introduced that makes this system a passive system for $c>0.18$. We have checked this numerically by computing the largest conditional Lyapunov exponent λ_1^c as a function of c. As can be seen in Fig. 5 λ_1^c is negative for $c>0.18$. This means that the response system becomes a passive system and generalized synchronization occurs [11], i.e., asymptotically for $t\rightarrow\infty$ the states of the response system are a function of the states of the drive system. Due to this strong relation the mean values of topological quantities like the rotation around a hole or singularity in the attractor have to be the same. The oscillations of the amplitudes may be linearly uncorrelated due to the possibly complicated structure of this nonlinear function, but the phase differences are bounded.

To check for which values of the coupling parameter c GS occurred in our experiment we applied the method of nearest neighbors [10,14] to detect the existence of a continuous function relating states of the drive to states of the response [15]. The states of drive and response were reconstructed from x_2 and y_2, respectively, in a three-dimensional state space with a delay of $t_l=0.02$. If the reconstructed states $\mathbf{u}^n=[y_2(t_n),y_2(t_{n-1}),y_2(t_{n-2})]$ of the response ($t_n=nt_l$) are given as a continuous function $\mathbf{h}(\mathbf{v}^n)$ of the states $\mathbf{v}^n=[x_2(t_n),x_2(t_{n-1}),x_2(t_{n-2})]$ of the drive, then any neighboring states of $\mathbf{u}^n$ are mapped to neighbors of $\mathbf{v}^n$. As a numerical indicator for the existence of a continuous function we have selected the nearest neigbor $\mathbf{u}^{nn}$ of $\mathbf{u}^n$ for $n=1,\ldots,N$ and have computed the average distance of the corresponding image points $\mathbf{v}^n$ and $\mathbf{v}^{nn}$. This mean distance of images of nearest neigbors was normalized by the average distance δ of randomly chosen states of the response system:

$$d=\frac{1}{N\delta}\sum_{n=1}^{N}\|\mathbf{v}^n-\mathbf{v}^{nn}\|. \tag{4}$$

The result of this continuity test is plotted in Fig. 6 vs the coupling constant c.

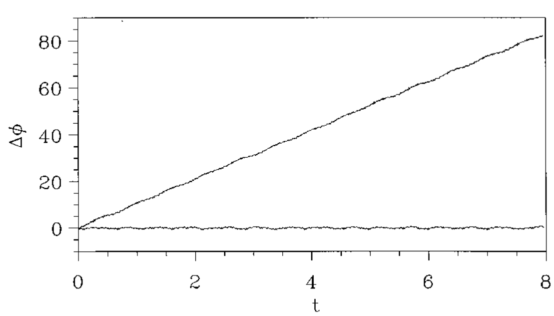

FIG. 3. Phase difference $\Delta\phi=\phi_1-\phi_2$ vs time for two representative cases: $c=0.1$, no PS; $c=0.2$, PS.

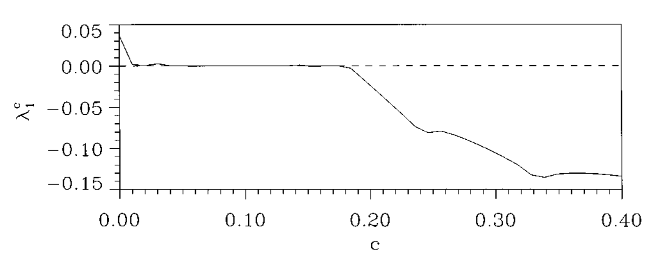

FIG. 5. Largest conditional Lyapunov exponent λ_1^c of the response system (3) vs coupling constant c. For $c>0.18$, $\lambda_1^c<0$ and the response system is passive.

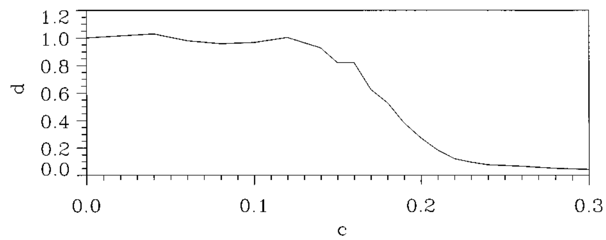

FIG. 6. Nearest neighbors test for GS applied to the experimental data. Plotted is the average distance d defined in Eq. (4) vs c. Near $c=0.18$ the quantity d decreases indicating the occurrence of GS.

As can be seen near $c=0.18$ a transition from 1 to 0 takes place indicating the occurrence of GS. This transition, however, is rather smooth compared to Fig. 2. We conjecture that this is due to the fact that the function **h** is in general quite complicated if the response system is only weakly passive, i.e., near the threshold value of the coupling [16]. When c is increased further the term $-cy_3$ in Eq. (3) leads to a more and more stable system and the function **h** becomes smoother. Of course, the smoother the function **h** the more close neighbors of the drive are mapped to close neighbors of the response. Tests for PS thus provide a more sensitive indicator for GS than nearest neighbors methods in those cases where they may be applied.

To conclude, we have presented an experimental observation of phase synchronization and generalized synchronization for a system of two unidirectionally coupled Rössler systems. In this case, a close relation between phase synchronization and generalized synchronization has been established with interesting consequences for both PS and GS. In general GS leads always to PS if one can define a suitable phase variable. On the other hand, PS may occur even in cases where the coupled systems show no GS, i.e., GS is the stronger property. We hope that these results stimulate further research on the details of the relation between GS, PS and other types of synchronization in uni-directionally as well as bidirectionally coupled systems.

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